

Quantum Coordinates of an Event in Local Quantum Physics

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Abstract

Recently [3] it has been proposed, using the formalism of positive-operator-valued measures, a possible definition of quantum coordinates for events in the context of quantum mechanics. In this short note we analyze this definition from the point of view of local algebras in the framework of local quantum theories.

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1 Introduction

Local quantum theory is now a well established framework for physical concepts and theories [1, 2]. A quite interesting problem is to see whether a given local quantum theory admits a sharp localization of a spacetime event; this problem is intrinsically related to the definition of spacetime at small scale and so to the quantum gravity problem.

In a recent work [3] Toller has proposed to define the localizability of an event using quantum coordinates via positive operator valued measures (POV-measures) [4]; such useful formalism has been used [5, 6, 7, 8] to investigate the longstanding problem [9, 10, 11, 12, 13, 14] of time in quantum mechanics and the quantum coordinates of [3] are a generalization of these works. In this short note we want to stress the fact that such quantum observable for the coordinates of an event cannot be built with quasilocal operators.

2 POV measures in the local framework

Let \mathcal{M} be the Minkowsky spacetime and $\mathcal{F}(\mathcal{M})$ be the σ -algebra of borel subsets of \mathcal{M} ; to the physical quantum object that defines the event is associated an Hilbert space \mathcal{H} and we call $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} and with $\mathcal{B}(\mathcal{H})^+$ the positive ones. A POV measure on \mathcal{M} with value in $\mathcal{B}(\mathcal{H})^+$ is a map

$$\tau : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{B}(\mathcal{H})^+$$

such that τ is σ -additive in the weak topology, $\tau(\emptyset) = 0$ and $\tau(\mathcal{M}) \leq 1$. If τ has to fix a quantum event we require that it is covariant with respect to a unitary representation of the Poincaré group. We remark the fact that we cannot, in general, require the normalization $\tau(\mathcal{M}) = 1$ for the POV-measure, as discussed in [3].

Let us consider now the formulation of a local quantum theory [2] in terms of local operators algebras; it is defined by a net of C^* -algebras, i.e. by an inclusion preserving assignment to every open, bounded region \mathcal{O} of \mathcal{M} of a unital C^* -algebra $\mathcal{A}(\mathcal{O})$; the closure in the norm topology of their union is the algebra of quasilocal observables

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathcal{M}} \mathcal{A}(\mathcal{O})}.$$

It is quite immediate to see that $\tau(\mathcal{I})$, with $\mathcal{I} \in \mathcal{F}(\mathcal{M})$ and $\mathcal{I} \neq \emptyset$, is not a local operator, i.e. it does not belong to one of the local algebras $\mathcal{A}(\mathcal{O})$. To see this we remind the reader that the vacuum state Ω is invariant under the translations on \mathcal{M} and so

$$(\Omega, \tau(\mathcal{I})\Omega) = (\Omega, U(x)\tau(\mathcal{I})U^{-1}(x)\Omega) = (\Omega, \tau(\mathcal{I}_x)\Omega)$$

where $U(x)$ is an unitary representation on \mathcal{H} of the group of spacetime translations and \mathcal{I}_x is the region \mathcal{I} transformed with the translation by x . If for \mathcal{I} we have

$$(\Omega, \tau(\mathcal{I})\Omega) = \epsilon$$

with $\epsilon > 0$, it does exist an integer n such that $n\epsilon > 1$. Now we can chose $n-1$ vectors $\{x_1, \dots, x_{n-1}\}$ such that $\mathcal{I} \cap \mathcal{I}_{x_i} = \emptyset$ and $\mathcal{I}_{x_i} \cap \mathcal{I}_{x_j} = \emptyset \quad \forall i, j \quad i \neq j$; for the additivity of τ and the invariance of the vacuum we have

$$(\Omega, \tau(\mathcal{I} \cup \mathcal{I}_{x_1} \cup \dots \cup \mathcal{I}_{x_{n-1}})\Omega) = n\epsilon > 1$$

that is impossible. So $(\Omega, \tau(\mathcal{I})\Omega) = 0$ and, since $\tau(\mathcal{I})$ is bounded and positive, for the Reeh-Schlieder [15] theorem it cannot be a local operator.

With a little more tricky demonstration we can see that it is neither a quasilocal operator. To see this we need the following trivial lemma

Lemma 1 *Given a POV measure τ on \mathcal{M} covariant with respect to a unitary representation of the translations group, if $\tau(\mathcal{I})$ is a quasi-local operator for some region \mathcal{I} then it is a quasi-local operator for every translated regions \mathcal{I}_x .*

A second necessary lemma is a standard result of local quantum physics (see [2, 16, 17]); let us consider n quasi-local operators

$$A_1 \dots A_n \in \mathcal{A}$$

and the automorphism of \mathcal{A} associated to a translation of x given by $\alpha_x A \equiv U(x)AU^{-1}(x)$. Let us divide the set of the indices $\{1 \dots n\}$ in two disjoint subsets I and J and define n new quasi-local operator $A'_1 \dots A'_n$ by

$$\begin{aligned} A'_i &= A_i \quad i \in I \\ A'_i &= A_i(x) \quad i \in J \end{aligned}$$

where $A_i(x) = \alpha_x A_i$. If we indicate $x = (t, \mathbf{x})$ then we have

Lemma 2 (Cluster property) *For $|\mathbf{x}| \rightarrow \infty$*

$$(\Omega, A'_1 \dots A'_n \Omega) \rightarrow (\Omega, \prod_{i \in I} A'_i \Omega) \cdot (\Omega, \prod_{j \in J} A'_j \Omega).$$

With these lemmas we can now demonstrate the following

Proposition 1 *If τ is a POV measure on \mathcal{M} covariant with respect to the spacetime translations group, then $\tau(\mathcal{I}) \notin \mathcal{A}$ for all $\mathcal{I} \in \mathcal{F}(\mathcal{M})$ with $\tau(\mathcal{I}) \neq 0$.*

Proof

Let us suppose that $\tau(\mathcal{I})$ is a quasi-local operator and let ψ be a normalized vector such that

$$(\psi, \tau(\mathcal{I})\psi) = p$$

with $0 < p \leq 1$; since the subset of vectors obtained by applying quasi-local operators to Ω is dense in \mathcal{H} one can assume that

$$\psi = A\Omega$$

where A is a quasi-local operator. Defining the vector $\psi_x = A(x)\Omega$, where $x = (t, \mathbf{x})$, for the covariance property of τ we have

$$(\psi_x, \tau(\mathcal{I}_x)\psi_x) = p.$$

We can now show that it exists a vector sequence $\{\tilde{\psi}_x\}$ such that, for $|\mathbf{x}| \rightarrow \infty$,

$$(\tilde{\psi}_x, \tau(\mathcal{I} \cup \mathcal{I}_x)\tilde{\psi}_x) \rightarrow 2p$$

and this is obviously impossible since $\tau(\mathcal{M}) \leq 1$.

In fact let us define the normalized vector

$$\tilde{\psi}_x = \frac{A(x)A\Omega}{\|A(x)A\Omega\|};$$

using the clustering lemma one can see that

$$\|A(x)A\Omega\|^2 \rightarrow \|A(x)\Omega\|^2 \cdot \|A\Omega\|^2 = 1$$

and that

$$(\tilde{\psi}_x, \tau(\mathcal{I})\tilde{\psi}_x) \rightarrow (\Omega, A^*(x)A(x)\Omega) \cdot (\Omega, A^*\tau(\mathcal{I})A\Omega) = (\psi, \tau(\mathcal{I})\psi) = p$$

for $|\mathbf{x}| \rightarrow \infty$. Similarly, in the same limit,

$$(\tilde{\psi}_x, \tau(\mathcal{I}_x)\tilde{\psi}_x) \rightarrow (\Omega, A^*A\Omega) \cdot (\Omega, A^*(x)\tau(\mathcal{I}_x)A(x)\Omega) = (\psi_x, \tau(\mathcal{I}_x)\psi_x) = p.$$

So

$$(\tilde{\psi}_x, \tau(\mathcal{I} \cup \mathcal{I}_x)\tilde{\psi}_x) = (\tilde{\psi}_x, \tau(\mathcal{I})\tilde{\psi}_x) + (\tilde{\psi}_x, \tau(\mathcal{I}_x)\tilde{\psi}_x) \rightarrow 2p.$$

If $2p > 1$ the demonstration is over; otherwise we can repeat the same argument finding normalized states such that the preceding probability tends toward $4p$ and so on.

Q.E.D.

So we see that the existence of a normalized POV measure covariant with respect to a unitary representation of the group of translations on \mathcal{M} is incompatible with the local principle if we require that $\tau(\mathcal{I})$ is quasi-local.

3 Conclusions

The use of POV-measures as observables is motivated by some deep considerations for the foundations of quantum mechanics [18, 19]. In particular it is very interesting the use of positive bounded operators, not necessarily projectors, as generalized propositions; it is an open problem to see what kind of observables it is possible to build with them. In this letter we have shown that a localizability for events in spacetime cannot be described by one of these observables if the generalized propositions have to belong to the algebra of quasilocal operators; further investigations are necessary in this direction.

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References

- [1] R.Haag and D.Kastler, *An Algebraic Approach to Quantum Field Theory*, Journ.Math.Phys. **5**, 848 (1964).
- [2] R.Haag, *Local Quantum Physics, 2nd. edition*, Springer-Verlag Berlin Heidelberg (1996).
- [3] M.Toller, *On the Quantum Space-Time Coordinates of an Event*, preprint quant-ph/9702060 (1997).
- [4] P.Busch, P.J.Lathi and P.Mittelstaedt *The Quantum Theory of Measurement*, Springer-Verlag (1991).
- [5] A.S.Holevo *Probabilistic and Statistical Aspect of Quantum Theory*, North-Holland (1982).

- [6] P.Busch, M.Grabowski and P.J.Lathi *Time Observables in Quantum Theory*, Phys.Lett. **A 191**, 357 (1994).
- [7] R.Giannitrapani *Positive-Operator-Valued Time Observable in Quantum Mechanics*, Int.Journ.Theor.Phys. **36**, 1575 (1997).
- [8] H.Atmianspacher and A.Amann *Positive Operator Valued Measures and Projector Valued Measures of Non-Commutative Time Operators*, preprint (1997).
- [9] N.Grot, C.Rovelli e R.S.Tate *Time of Arrival in Quantum Mechanics*, Phys.Rev.A **54**, 4676 (1996).
- [10] V.Delgado and J.G.Muga *Arrival Time in Quantum Mechanics*, Phys.Rev. **A56** 3425 (1997).
- [11] V.Delgado *Probability Distribution of Arrival Times in Quantum Mechanics*, preprint quant-ph/9709037 (1997).
- [12] J.Leon *Time of Arrival Formalism for the Relativistic Particle*, J. Phys. **A30** 4791 (1997).
- [13] J.G.Muga, R.Sala e J.P.Palao *The Time of Arrival Concept in Quantum Mechanics*, in Superlattices and Microstructures **24** (1998).
- [14] Y.Aharonov, J.Oppenheim, S.Popescu, B.Reznik and W.G.Unruh *Measurement of Time-of-Arrival in Quantum Mechanics*, preprint quant-ph/9709031 (1997).
- [15] H.Reeh and S.Schlieder, *Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern*, Nuovo Cimento **22**, 1051 (1961).
- [16] R.Jost, *The General Theory of Quantized Fields*, Lectures in Applied Mathematics (1964).
- [17] R.F.Streater and A.S.Wightman, *PCT, Spin and Statistics, and all that*, Mathematical Physics Monograph Series (1977).
- [18] R.Giles, *Foundations for Quantum Mechanics*, Journ.Math.Phys. **11**, 2139 (1970).
- [19] G.Ludwig, *Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories III*. Commun.Math.Phys. **9**, 1 (1968).